Capacitary representations of positive solutions of semilinear parabolic equations

Moshe Marcus, Laurent Veron

CAPACITARY REPRESENTATION OF POSITIVE SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS

MOSHE MARCUS AND LAURENT VERON

ABSTRACT. We give a global bilateral estimate on the maximal solution $\bar{u}_F$ of $\partial_t u - \Delta u + u^q = 0$ in $\mathbb{R}^N \times (0, \infty)$, $q > 1$, $N \geq 1$, which vanishes at $t = 0$ on the complement of a closed subset $F \subset \mathbb{R}^N$. This estimate is expressed by a Wiener test involving the Bessel capacity $C_{2/q,q'}$. We deduce from this estimate that $\bar{u}_F$ is σ-moderate in Dynkin's sense.

Représentation capacitaire des solutions positives d'équations paraboliques semi-linéaires

RÉSUMÉ. Nous donnons une estimation bilatérale précise de la solution maximale $\bar{u}_F$ de $\partial_t u - \Delta u + u^q = 0$ dans $\mathbb{R}^N \times (0, \infty)$, $q > 1$, $N \geq 1$, qui s'annulle en $t = 0$ sur le complémentaire d'un sous-ensemble fermé $F \subset \mathbb{R}^N$. Cette estimation s'exprime par un test de Wiener impliquant la capacité de Bessel $C_{2/q,q'}$. Nous déduisons de cette estimation que $\bar{u}_F$ est σ-moderée au sens de Dynkin.

VERSION FRANÇAISE ABRÉGÉE

Soit $q > 1$. Si u est une solution positive de

$$\partial_t u - \Delta u + |u|^{q-1} u = 0 \tag{1}$$

dans $\mathbb{R}^N \times (0, \infty)$, nous avons démontré dans [6] qu'elle admet une trace initiale, notée $Tr(u)$, dans la classe des mesures de Borel positives et régulières, mais pas nécessairement localement bornées. Si F est un sous-ensemble fermé de $\mathbb{R}^N$, nous désignons par $\bar{u}_F$ la solution maximale de (1) dont le support de la trace initiale est inclus dans F. Si $1 < q < q_c := (N + 2)/N$, il est montré dans [5] que les inégalités suivantes sont vérifiées

$$t^{-1/(q-1)} f(|x - a| / \sqrt{t}) \leq \bar{u}_F(x, t) \leq ((q - 1)t)^{-1/(q-1)} \quad \forall a \in F,$$

où f est l'unique fonction positive vérifiant

$$\Delta f + \frac{1}{2} y \cdot Df + \frac{1}{q-1} f - |f|^{q-1} f = 0 \text{ dans } \mathbb{R}^N \text{ et } \lim_{|y| \to \infty} |y|^{2/(q-1)} f(y) = 0.$$

Ces inégalités jouent un rôle fondamental dans la démonstration du caractère biunivoque de la correspondance, par l'opérateur de trace initiale, entre l'ensemble des solutions positives de (1) et l'ensemble des mesures de Borel positives régulières. Quand $q \geq q_c$ la fonction f est identiquement nulle car les singularités isolées de (1) sont éliminables [4].

Définition. *Soit $N \geq 1$, $q \geq q_c$ et F un sous-ensemble fermé de $\mathbb{R}^N$. On définit le potentiel $(2/q, q')$-capacitaire W_F de F par*

$$W_F(x,t) = t^{-1/(q-1)} \sum_{n=0}^{\infty} (n+1)^{N/2-1/(q-1)} e^{-n/4} C_{2/q,q'} \left(\frac{F_n}{\sqrt{(n+1)t}} \right), \tag{2}$$

$\forall (x,t) \in \mathbb{R}^N \times [0,\infty)$, où $F_n = F_n(x,t) = \left\{ y \in F : \sqrt{nt} \leq |x-y| < \sqrt{(n+1)t} \right\}$.

Notre résultat principal est l'estimation bilatérale.

Théorème 1. *Soit $N \geq 1$, $q \geq q_c$. Il existe deux constantes $C_1 > C_2 > 0$, ne dépendant que de N et q, telles que pour tout sous-ensemble fermé F de $\mathbb{R}^N$*

$$C_2 W_F(x,t) \leq \bar{u}_F(x,t) \leq C_1 W_F(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{3}$$

Si $\mu \in \mathfrak{M}^b(\mathbb{R}^N)$ (l'espace des mesures de Radon bornées dans $\mathbb{R}^N$) appartient à $W^{-2/q,q}(\mathbb{R}^N)$, il existe une unique solution $u = u_\mu$ de (1) dont la trace initiale est μ [3]. On définit alors

$$\underline{u}_F = \sup\{u_\mu : \mu \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N) : \mu(F^c) = 0\}. \tag{4}$$

Cette solution est σ-modérée au sens de Dynkin, c'est à dire qu'il existe une suite croissante de mesures positives $\mu_n \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N)$ telles que $u_{\mu_n} \uparrow \underline{u}_F$.

La clef de la démonstration du Théorème 1 est l'estimation inférieure de $\underline{u}_F$.

Théorème 2. *Soit $N \geq 1$, $q \geq q_c$. Il existe une constante $C = C(N,q) > 0$ telle que pour tout sous-ensemble fermé F de $\mathbb{R}^N$,*

$$\underline{u}_F(x,t) \geq C W_F(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{5}$$

Une conséquence importante des estimations précédentes est la suivante:

Théorème 3. *Soit $N \geq 1$, $q > 1$. Pour tout sous-ensemble fermé F de $\mathbb{R}^N$, $\bar{u}_F = \underline{u}_F$.*

1. Main results

Let u be a nonnegative solution of

$$\partial_t u - \Delta u + |u|^{q-1} u = 0, \qquad q > 1 \tag{1}$$

in $\mathbb{R}^N \times (0,\infty)$. It was proved in [6] that u admits an initial trace, denoted by $Tr(u)$, in the class of outer regular positive Borel measures, not necessarily locally bounded. If F is a closed subset of $\mathbb{R}^N$, we denote by $\bar{u}_F$ the maximal solution of (1) which belongs to $C(F^c \times [0,\infty))$ and vanishes on $F^c \times \{0\}$, where $F^c := \mathbb{R}^N \setminus F$.

If $1 < q < (N+2)/N$, the following inequalities are verified

$$t^{-1/(q-1)} f(|x-a| / \sqrt{t}) \leq \bar{u}_F(x,t) \leq ((q-1)t)^{-1/(q-1)} \quad \forall a \in F, \tag{2}$$

where f is the unique positive solution [5] of

$$\Delta f + \frac{1}{2} y \cdot Df + \frac{1}{q-1} f - |f|^{q-1} f = 0 \ \text{ in } \mathbb{R}^N \ \text{ s. t. } \lim_{|y| \to \infty} |y|^{2/(q-1)} f(y) = 0. \tag{3}$$

These inequalities play a fundamental role in proving that (in the subcritical case) any positive solution is uniquely determined by its initial trace [6].

Definition 1. *Let*

$$B_{\sqrt{nt}}(x) = \{y \in \mathbb{R}^N : |x-y| < \sqrt{nt}\}$$
$$T_n(x,t) = \bar{B}_{\sqrt{(n+1)t}}(x) \setminus B_{\sqrt{nt}}(x). \tag{4}$$

For every $q \geq q_c$ and every closed subset $F \subset \mathbb{R}^N$, we define the $(2/q, q')$-capacitary potential W_F of F by

$$W_F(x,t) = t^{-1/(q-1)} \sum_{n=0}^{\infty} (n+1)^{N/2-1/(q-1)} e^{-n/4} C_{2/q,q'}\Big(\frac{F \cap T_n(x,t)}{\sqrt{(n+1)t}}\Big), \tag{5}$$

$\forall (x,t) \in \mathbb{R}^N \times [0,\infty)$.

Our main result is the following bilateral estimate

Theorem 1. *Let $N \geq 1$, $q \geq q_c$. There exist two positive constants C_1 and C_2, depending on N and q such that for any closed subset F of $\mathbb{R}^N$,*

$$C_2 W_F(x,t) \leq \bar{u}_F(x,t) \leq C_1 W_F(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{6}$$

Remark. It is important to notice that although W_F is not a solution of (1), it is invariant with respect to the similarity transformation associated with this equation:

$$k^{1/(q-1)} W_F(\sqrt{k}x, kt) = W_{F/\sqrt{k}}(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty),\ \forall k > 0. \tag{7}$$

Clearly $\bar{u}_F$ is also self similar, i.e., invariant with respect to the above transformation.

If μ is a bounded Borel measure such that $|\mu| \in W^{-2/q,q}(\mathbb{R}^N)$ then, there exists a unique solution $u = u_\mu$ of (1) with initial trace μ. We define

$$\underline{u}_F = \sup\{u_\mu : \mu \in W_+^{-2/q,q}(\mathbb{R}^N),\ \mu(F^c) = 0\}.$$

This solution is σ-moderate in the sense of Dynkin, which means that there exists an increasing sequence of positive measures $\mu_n \in W_+^{-2/q,q}(\mathbb{R}^N)$ such that $u_{\mu_n} \uparrow \underline{u}_F$. Clearly $\underline{u}_F \leq \bar{u}_F$. Therefore the next result implies the lower estimate in Theorem 1.

Theorem 2. *Let $N \geq 1$, $q \geq q_c$. There exists a positive $C = C(N,q)$, such that for any closed subset F of $\mathbb{R}^N$,*

$$\underline{u}_F(x,t) \geq C W_F(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{8}$$

As a consequence of Theorems 1 and 2 we find that $\bar{u}_F \leq c\underline{u}_F$. Using this fact we obtain the following result (already known [6] in the case $1 < q < q_c$),

Theorem 3. *Let $N \geq 1$, $q > 1$. For any closed subset F of $\mathbb{R}^N$ one has $\bar{u}_F = \underline{u}_F$. In particular $\bar{u}_F$ is σ-moderate.*

2. Proof of the upper estimate in Theorem 1

In the sequel we denote by c a positive constant which depends only on N and q; its value may change from one occurrence to another. Without loss of generality, we assume that F is compact. Denote $B_r(x) = \{y : |x-y| < r\}$ and $B_r = B_r(0)$.

Let $r > 0$ be a positive number such that $F \subset B_r$. We start by deriving an upper estimate depending on r. Let $\rho > 0$ be a positive number, to be later determined as a function of r, t. Let $\eta \in C_0^\infty(B_{r+\rho})$ be such that $\eta = 1$ on F and $0 \leq \eta \leq 1$. Put $\eta^* = 1 - \eta$ and choose $\zeta := \left(e^{t\Delta}[\eta^*]\right)^{2q'}$ as a test function. Then

$$\int_0^1 \int_{\mathbb{R}^N} u(\partial_t - \Delta)\zeta dxdt + \int_0^1 \int_{\mathbb{R}^N} u^q \zeta dxdt = -\int_{\mathbb{R}^N} u(x,1)dx. \tag{9}$$

A straightforward computation yields

$$\int_{Q_{r+\rho}} u^q dxdt + \int_{\mathbb{R}^N} u(x,1)dx \leq \int_0^1 \int_{\mathbb{R}^N} (R(\eta))^{q'}\, dx\, dt,$$

$$R(\eta) := \left|De^{t\Delta}[\eta]\right|^2 + \left|\partial_t e^{t\Delta}[\eta] + \Delta e^{t\Delta}[\eta]\right|, \quad Q_r := \{(x,t) : t > 0,\ |x|^2 + t \geq r^2\}.$$

Using interpolation inequalities [8] one obtains,

$$\int_0^1 \int_{\mathbb{R}^N} (R(\eta))^{q'}\, dx\, dt \leq C_1 \|\eta\|^{q'}_{W^{2/q,q'}}. \tag{10}$$

This implies

$$\int_{\mathbb{R}^N} u(x,1)\, dx + \int_{Q_{r+\rho}} u^q\, dx\, dt \leq c C^{B_{r+\rho}}_{2/q,q'}(F).$$

Further it can be shown that, for $0 < s < 1$,

$$\int_{\mathbb{R}^N} u(x,1)\, dx + \int_s^1 \int_{\mathbb{R}^N} u^q\, dx\, dt = \int_{\mathbb{R}^N} u(x,s)\, dx. \tag{11}$$

Clearly $u(x, t+s) < w_s(x,t) := e^{t\Delta}[u(\cdot, s)]$ for $t > 0$. Therefore, by (10) and (11),

$$u(x,(r+2\rho)^2) \leq \frac{c}{(\rho^2 + r\rho)^{N/2}} C^{B_{r+\rho}}_{2/q,q'}(F). \tag{12}$$

Let v be the solution of the initial-boundary value problem

$$\partial_t v - \Delta v = 0 \ \text{ in } \ Q^*_{r,\rho} := (\bar{B}_{r+\rho})^c \times (0,(r+2\rho)^2),$$
$$v(x,0) = 0 \quad \forall x \in B^c_{r+\rho}, \quad v = u \quad \forall (x,t) \in \partial B_{r+\rho} \times (0,(r+2\rho)^2).$$

Using the fact that $u < v$ in $Q^*_{r,\rho}$ and (12) we obtain,

Lemma 4. *If $F \subset B_r$, there exists $c > 0$ such that*

$$\bar{u}_F(x,t) \leq c\left(1 + \frac{r}{\rho}\right)^{N/2} \frac{e^{-(|x|-r-3\rho)^2/4t}}{t^{N/2}} C^{B_{r+\rho}}_{2/q,q'}(F), \tag{13}$$

for any $(x,t) \in \mathbb{R}^N \times [(r+3\rho)^2, \infty)$.

The upper estimate in (6) is obtained by slicing F, relative to a given point $(x,t) \in \mathbb{R}^N \times (0,\infty)$, in such a way that each slice satisfies the assumption of Lemma 4 for an appropriate value of r depending on the point. Put

$$T_n(x,t) = B_{\sqrt{(n+1)t}}(x) \setminus B_{\sqrt{nt}}(x),\ F_n(x,t) = F \cap T_n(x,t) \quad \forall n \in \mathbb{N}.$$

Since a sum of positive solutions of (1) is a super-solution,

$$\bar{u}_F \leq \sum_{n=0}^{\infty} \bar{u}_{F_n(x,t)}.$$

Using this fact and Lemma 4 we show that

$$\bar{u}_F(x,t) \leq c W_F(x,t) \tag{14}$$

for every $(x,t)\in\mathbb{R}^N\times(0,\infty)$. In view of the fact that both sides of this inequality are invariant with respect to the similarity transformation (7), it is sufficient to prove it in the case $(x,t)=(0,1)$. We denote $F_n=F_n(0,1)$ and $T_n=T_n(0,1)$.

If $N=1$, each of the sets F_n satisfies the condition of Lemma 4 with $r=\sqrt{n}$. But this is not the case when $N\geq 2$. Therefore, if $N\geq 2$, a secondary slicing is needed.

For every $n\in\mathbb{N}$ there exists a set of points $\Theta_n=\{a_{j,n}\}_{j=1}^{J_n}$ on the sphere $|y|=(\sqrt{n+1}+\sqrt{n})/2$ such that

$$|a_{n,j}-a_{n,k}|\geq 1/\sqrt{2(n+1)}\ \text{ for }\ j\neq k,\quad T_n\subset\bigcup_{1\leq j\leq J_n}B_{\sqrt{1/(n+1)}}(a_{n,j}).$$

Clearly $J_n\leq(\sqrt{2}(n+1))^{N-1}<(4n)^{N-1}$. If $F_{n,j}:=F_n\cap B_{\sqrt{1/(n+1)}}(a_{n,j})$,

$$\bar{u}_F(0,1)\leq\sum_{n=0}^{\infty}\sum_{1\leq j\leq J_n}\bar{u}_{F_{n,j}}(0,1). \tag{15}$$

It is not difficult to verify that

$$C_{2/q,q'}^{B_{2/\sqrt{n+1}}(a_{n,j})}(F_{n,j})\approx(n+1)^{N/2-1/(q-1)}C_{2/q,q'}\left(\sqrt{n+1}\,F_{n,j}\right)$$

where the capacity on the right hand side (resp. left hand side) is the Bessel capacity relative to $\mathbb{R}^N$ (resp. relative to $B_{2/\sqrt{n+1}}(a_{n,j})$). The symbol $\approx$ stands for two-sided inequalities with constants $c=c(N,q)$. Further, the quasi-additivity of Bessel capacities [2] implies

$$\sum_{1\leq j\leq J_n}C_{2/q,q'}(\sqrt{n+1}\,F_{n,j})\leq c(N,q)\,C_{2/q,q'}(\sqrt{n+1}\,F_n). \tag{16}$$

Each set $F_{n,j}$ satisfies the condition of Lemma 4 with $r=\sqrt{n}$. Therefore, estimating $\bar{u}_{F_{n,j}}(0,1)$ as in (13) and using (15) and (16) we obtain (14) for $(x,t)=(0,1)$.

3. Proof of Theorem 2

As in the elliptic case [7], for each $(x,t)\in\mathbb{R}^N\times(0,\infty)$, we construct a measure $\mu=\mu_{x,t}\in W_+^{-2/q,q}(\mathbb{R}^N)$, concentrated on F, such that

$$u_\mu(x,t)\geq cW_F(x,t). \tag{17}$$

Since $\underline{u}_F>u_\mu$, (17) implies (8).

For every measure $\mu\in W_+^{-2/q,q}(\mathbb{R}^N)$, $0\leq u_\mu\leq e^{t\Delta}[\mu]$. Therefore

$$u_\mu\geq e^{t\Delta}[\mu]-\int_0^t e^{(t-s)\Delta}[(e^{s\Delta}[\mu])^q]ds. \tag{18}$$

Let ν_n be the capacitary measure of $F_n(x,t)/\sqrt{t(n+1)}$ (see [1]) and define the measure μ_n by

$$\mu_n(A)=(t(n+1))^{N/2-1/(q-1)}\nu_n(A/\sqrt{t(n+1)}),$$

for every Borel set A. Thus μ_n is concentrated on F_n. Finally put $\mu:=\mu_{x,t}=\sum_n\mu_n$. With this choice of μ it is not difficult to show that

$$e^{t\Delta}[\mu](x,t)\geq\frac{1}{(4\pi t)^{N/2}}\sum_{n=0}^{\infty}(\sqrt{(n+1)t})^{N-2/(q-1)}e^{-(n+1)/4}C_{2/q,q'}\Big(\frac{F_n(x,t)}{\sqrt{(n+1)t}}\Big). \tag{19}$$

We have to derive a corresponding upper estimate for the nonlinear term in (18). To this end we employ a partitioning $\{\mathcal{T}_n : n \in \mathbb{Z}\}$ of $\mathbb{R}^N \times (0,t)$ defined by,

$$\mathcal{T}_n = \begin{cases} \{(y,s) : tn \leq |x-y|^2 + t - s \leq t(n+1),\ 0 < s < t\}, & \text{if } n \in \mathbb{N}_*, \\ \{(y,s) : t\alpha^{-n} \leq |x-y|^2 + t - s \leq t\alpha^{-n-1},\ 0 < s < t\}, & \text{if } n \leq 0, \end{cases}$$

where $\alpha \in (0,1)$ must be appropriately chosen. If $G(\xi,\tau) := (4\pi|\tau|)^{-N/2}\exp(-|\xi|^2/4\tau))$ then

$$\int_0^t e^{-(t-s)\Delta}(e^{s\Delta}[\mu])^q ds = C\sum_{p\in\mathbb{Z}}\iint_{\mathcal{T}_p} G(y-x,s-t)\Big(\sum_{n=0}^{\infty}\int_{\mathbb{R}^N} G(z-y,s)d\mu_n(z)\Big)^q dyds.$$

We denote

$$J_1 = \sum_{p\in\mathbb{Z}}\iint_{\mathcal{T}_p} G(y-x,s-t)\Big(\sum_{n=0}^{p+2}\int_{\mathbb{R}^N} G(z-y,s)s^{N/2}d\mu_n(z)\Big)^q dyds,$$

$$J_2 = \sum_{p\in\mathbb{Z}}\iint_{\mathcal{T}_p} G(x-y,s-t)\Big(\sum_{n=p+3}^{\infty}\int_{\mathbb{R}^N} G(z-y,s)s^{N/2}d\mu_n(z)\Big)^q dyds.$$

J_1 is estimated using Hölder's inequality and the following inequalities [8]

$$\frac{1}{C}\|\lambda\|_{W^{-2/q,q}} \leq \left\|e^{t\Delta}[\lambda]\right\|_{L^q(\mathbb{R}^N\times(0,1))} \leq C\|\lambda\|_{W^{-2/q,q}}, \tag{20}$$

valid for any $\lambda \in W^{-2/q,q}$. The estimate of J_2, in the case $N > 1$, requires a more delicate argument using the secondary slicing introduced in the previous section and the quasi-additivity of Bessel capacities. For a suitable choice of α one obtains

$$J_1 + J_2 \leq \frac{C}{t^{N/2}}\sum_{n=0}^{\infty}(\sqrt{(n+1)t})^{N-2/(q-1)}e^{-(n+1)/4}C_{2/q,q'}\Big(\frac{F_n}{\sqrt{(n+1)t}}\Big). \tag{21}$$

The inequalities (18), (19) and (21) imply (17), with μ replaced by $\epsilon\mu$, provided that $\epsilon > 0$ is sufficiently small, depending on q and the constants in (19) and (21). This in turn implies (8).

It follows that

$$\underline{u}_F(x,t) \leq \bar{u}_F(x,t) \leq c\underline{u}_F(x,t).$$

By the uniqueness argument used in [6] we conclude that $\bar{u}_F(x,t) = \underline{u}_F(x,t)$.

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Department of Mathematics, Technion, Haifa, ISRAEL

Laboratoire de Mathématiques, Faculté des Sciences, Parc de Grandmont, 37200 Tours, FRANCE